THE LIMIT POINTS OF A GROUP*

BY

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1. **Introduction.** The present paper is concerned with properly discontinuous groups of linear transformations. Such a group can be so transformed that the point at infinity is carried into some fixed finite region by every transformation of the group other than the identical transformation. In what follows it will be understood, without express statement, that the groups referred to have this property.

The *isometric circle*† of a linear transformation

$$z' = T(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad c \neq 0,$$

is the circle

$$|cz + d| = 1.$$

The isometric circles exist for all transformations of a group other than the identical transformation. For a given group these circles all lie in a finite region and have distinct centers; and they have the property that there is only a finite number with radius exceeding a given positive quantity.

By the *limit points* of a group is meant the cluster points of the centers of its isometric circles. It follows from the preceding remarks that there exist isometric circles belonging to the transformations of the group which lie wholly within any circle, however small, drawn with center at a limit point.

A group is called *elementary* if it has no limit points, or one limit point, or two limit points. If a group has more than two limit points, its limit points form a perfect set and so are non-denumerably infinite in number. It is with these latter, or non-elementary, groups that we shall be concerned.

2. **The fundamental theorem.** In the study of the limit points we shall make use of the properties of the isometric circles. The various results are based on the following theorem.

THEOREM 1. *Let C and K be circles whose centers are the limit points P and Q respectively of a non-elementary group. Then there exists a transformation of the group whose isometric circle lies in C and the isometric circle of whose inverse lies in K.*

Let

$$I_1, I_2, I_3, \cdots$$

be a sequence of isometric circles belonging to the group and lying in C with centers

$$g_1, g_2, g_3, \cdots \rightarrow P \; ;$$

and let

$$J_1, J_2, J_3, \cdots$$

be a sequence lying in K with centers

$$h_1, h_2, h_3, \cdots \rightarrow Q.$$

Let T_n and S_n be the transformations of the group whose isometric circles are I_n and J_n respectively.

Paired with each isometric circle is the circle of equal radius belonging to the inverse transformation. Thus T_n carries I_n into I_n', the isometric circle of T_n^{-1}. Let

$$I_1', I_2', \cdots \; ; \qquad J_1', J_2', \cdots$$

represent the isometric circles of the inverses and let the centers of these be denoted by

$$g_1', g_2', \cdots \; ; \qquad h_1', h_2', \cdots .$$

Let G' be the set of cluster points of the set g_n', and H' the set of cluster points of the set h_n'. Each set has at least one cluster point, since it consists of an infinite set of points lying in a finite region.

We shall consider first the case $P \neq Q$. We may suppose (decreasing the circles, if necessary) that C and K are external to one another.

(1) Suppose that G' contains a point g' and H' a point h' where $g' \neq h'$. We can select subsequences of g_n' and h_n' approaching g' and h' respectively:

$$g_{n_1}', g_{n_2}', \cdots \rightarrow g' \; ; \; h_{m_1}', h_{m_2}', \cdots \rightarrow h'.$$

We shall make repeated use of the following lemma, which has been proved elsewhere.*

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* Proceedings of the National Academy of Sciences, vol. 13 (1927), p. 290; also *Automorphic Functions*, §24.

LEMMA. *Let I_t, I'_t, I_s, I'_s, and I_u be the isometric circles of T, T^{-1}, S, S^{-1}, and $U = TS$ respectively. If I_t and I'_s are exterior to one another then I_u is contained in I_s.*

Consider the transformation

$$T = S^{-1}_{m_k} T_{n_k} ; \quad T^{-1} = T^{-1}_{n_k} S_{m_k}.$$

For k sufficiently large the isometric circles of $S_{m_k}^{-1}$ and $T_{n_k}^{-1}$, namely J'_{m_k} and I'_{n_k}, are arbitrarily near h' and g' respectively and so are external. It follows from the lemma that the isometric circle of T lies in I_{n_k}, and hence in C. Similarly, applying the lemma to T^{-1}, the isometric circle of T^{-1} lies in J_{m_k} and in K. This proves the theorem for this case.

(2) Points g' and h' for the preceding treatment exist unless G' and H' each consists of a single point: $G' = H' = \alpha$.

If $\alpha = P$ or $\alpha = Q$ the theorem obviously holds for the transformation S_n or T_n for n sufficiently large. So we shall assume $\alpha \neq P$, $\alpha \neq Q$.

By taking n and m large enough we can make I_n as near to P, J_m as near to Q, and I'_n and J'_m as near to α as we wish. Hence we can choose n and m so that I_n and I'_n are exterior to one another and J_m and J'_m are exterior to one another. Now when the isometric circles of a transformation and its inverse are external circles the transformation is loxodromic or hyperbolic and each circle contains one of the fixed points of the transformation. Then I'_n contains a fixed point α_n of T_n and J'_m contains a fixed point β_m of S_m.

Suppose $\alpha_n \neq \beta_m$. Now α_n is a fixed point of T_n^q and β_m is a fixed point of S_m^q. If $q > 1$ the isometric circle of T_n^q lies in I_n and that of S_m^q lies in J_m. Also the isometric circle of T_n^{-q} contains α_n and that of S_m^{-q} contains β_m. By taking q large enough we can make the radii of these circles less than $|\alpha_n - \beta_m|/4$. Then the isometric circles of T_n^{-q} and S_m^{-q} are external circles.

We now apply the lemma to the transformation $T = S_m^{-q} T_n^q$. Its isometric circle lies in that of T_n^q, and hence in I_n and in C. Similarly, the isometric circle of the inverse, $T^{-1} = T_n^{-q} S_m^q$, lies in that of S_m^q, and so in J_m and in K.

We now show that $\alpha_n = \beta_m$ is impossible for n and m large enough. The remaining fixed points of T_n and S_m are distinct since one lies near P and the other near Q. That $\alpha_n \neq \beta_m$ then follows directly from the following proposition.

If two hyperbolic or loxodromic transformations belonging to a group have one fixed point in common but the other fixed points distinct, then the group contains infinitesimal transformations.

Let the group be so transformed that the common fixed point is at infinity. Then the transformations have the forms

$$T: \quad z' - a = M(z - a), \quad |M| \neq 1,$$
$$S: \quad z' - b = N(z - b), \quad |N| \neq 1, \quad b \neq a;$$

and the powers of these transformations are

$$T^n(z) = M^n z + a(1 - M^n),$$
$$S^m(z) = N^m z + b(1 - N^m).$$

We now form the following product:

$$U = T^{-n}S^{-m}T^nS^m.$$

We find, on working this out, the translation

$$U = z + \omega,$$

where

$$\omega = (a - b)(1 - M^{-n})(1 - N^{-m}).$$

By suitable choices of m and n we can make M^{-n} and N^{-m} arbitrarily small and so make the period ω arbitrarily near $a - b$.

Given $\epsilon > 0$, we form two such translations

$$U_1 = z + a - b + \epsilon_1, \quad U_2 = z + a - b + \epsilon_2,$$

where $|\epsilon_1| < \frac{1}{2}\epsilon$, $|\epsilon_2| < \frac{1}{2}\epsilon$, and $\epsilon_1 \neq \epsilon_2$. Then

$$U_1 U_2^{-1} = z + \epsilon_1 - \epsilon_2$$

is a translation of period $\epsilon_1 - \epsilon_2$, where $0 < |\epsilon_1 - \epsilon_2| < \epsilon$. Since ϵ may be chosen arbitrarily small, the group contains infinitesimal transformations. It is therefore not a discontinuous group.

There remains the case $P = Q$. We wish to prove that there exists a transformation T whose isometric circle and the isometric circle of whose inverse both lie in the circle C surrounding P.

Let C be made small enough that it does not contain all the limit points, and let P_1 and P_2 be distinct limit points outside C. About P_1 and P_2 construct circles C_1 and C_2 exterior to one another and to C.

Let T_1 be a transformation such that its isometric circle I_1 lies in C and that of its inverse, I_1', lies in C_1, and let T_2 be a transformation such that its isometric circle I_2 lies in C and that of its inverse, I_2', lies in C_2. We now apply the lemma to the transformation

$$T = T_2^{-1}T_1, \qquad T^{-1} = T_1^{-1}T_2.$$

Since I_2' and I_1' are exterior to one another the isometric circle of T lies in I_1 and hence in C. Similarly, the isometric circle of T^{-1} lies in I_2 and hence in C. This completes the proof of the theorem.

COROLLARY. *Let P and Q be limit points of a non-elementary group. Then there exists a transformation of the group, one of whose fixed points is in an arbitrarily small neighborhood of P and the other is in an arbitrarily small neighborhood of Q.*

This follows from the fact that the fixed points of a transformation lie within or on the isometric circles of the transformation and its inverse. In this corollary P and Q are not necessarily distinct.

We are now able to prove the following precise theorem embodying an important property of the limit points.

THEOREM 2. *Let C be a circle with center at a limit point P of a non-elementary group, and let Q be any point whatever in the plane. Then C contains an infinite number of distinct points congruent to Q.*

It suffices to prove that C contains one point congruent to Q and distinct from P; for we can then replace C by a circle C' with center P and small enough to exclude the first congruent point and thus get a second congruent point, and so step by step get an infinite number of distinct congruent points. Let P_1 be a limit point lying in C and distinct from P and Q (here Q and P may be identical). Such a point exists, since P is a cluster point of limit points. Let P_2 be any other limit point distinct from P and Q. Let C_1 and C_2 be circles with centers P_1 and P_2 respectively and sufficiently small that P and Q are exterior to both circles and that C_1 lies in C. Let T be a transformation whose isometric circle I lies in C_2 and the isometric circle I' of whose inverse lies in C_1 (Theorem 1). T carries the exterior of I into the interior of I'. Hence Q is carried into a point Q' lying in I'. Then Q' lies in C and is distinct from P; and the theorem is established.

This theorem is no longer true if we drop the requirement that the group be non-elementary. Its failure occurs in the case of an elementary group if Q is a limit point.

3. **Non-loxodromic groups.** A properly discontinuous group is called Fuchsian if the transformations have a common fixed circle and each carries the interior of the circle into itself. It is well known that a Fuchsian group contains no loxodromic transformation. That a group without loxodromic transformations is not necessarily Fuchsian is shown by the doubly periodic group. This is, however, an elementary group. We shall prove the following

THEOREM 3. *A non-elementary properly discontinuous group which contains no loxodromic transformations is a Fuchsian group.*

We shall show first that all the limit points lie on a circle. Let A, B, C be distinct limit points and let K be the circle determined by them. We shall

show that any other limit point P lies on K. The necessary and sufficient condition for this is that the cross ratio

$$\lambda = \frac{(A - B)(C - P)}{(A - C)(B - P)}$$

be real.

The transformations

$$T(z) = (az + b)/(cz + d), \quad ad - bc = 1,$$

$$S(z) = (\alpha z + \beta)/(\gamma z + \delta), \quad \alpha\delta - \beta\gamma = 1,$$

are non-loxodromic if and only if $a+d$ and $\alpha+\delta$ are real. Let I_t, I_t', I_s, I_s' be the isometric circles of T, T^{-1}, S, S^{-1} respectively. The centers of these circles are $-d/c$, a/c, $-\delta/\gamma$, α/γ respectively. The first pair have radius $1/|c|$, the second pair $1/|\gamma|$.

According to Theorem 1 we can find transformations T and S belonging to the group so that a/c, $-d/c$, α/γ, $-\delta/\gamma$ are arbitrarily near A, C, B, P respectively and so that $1/|c|$ and $1/|\gamma|$ are arbitrarily small. Given $\epsilon > 0$, we can choose T and S so that

$$\mu = \frac{\left(\dfrac{a}{c} - \dfrac{\alpha}{\gamma}\right)\left(-\dfrac{\alpha}{c} + \dfrac{\delta}{\gamma}\right)}{\left(\dfrac{a}{c} + \dfrac{d}{c}\right)\left(\dfrac{\alpha}{\gamma} + \dfrac{\delta}{\gamma}\right)} = \lambda + \epsilon'$$

where $|\epsilon'| < \frac{1}{2}\epsilon$, and also so that $|\epsilon''| < \frac{1}{2}\epsilon$, where

$$\epsilon'' = \frac{\dfrac{1}{c^2} + \dfrac{1}{\gamma^2}}{\left(\dfrac{a}{c} + \dfrac{d}{c}\right)\left(\dfrac{\alpha}{\gamma} + \dfrac{\delta}{\gamma}\right)} .$$

Now by hypothesis $S^{-1} T$ is non-loxodromic. We find readily that the condition for this is that

$$- \delta a + \beta c + \gamma b - \alpha d$$

be real. Hence

$$\nu = - \frac{- \delta a + \beta c + \gamma b - \alpha d}{(a + d)(\alpha + \delta)}$$

is real. On dividing by γc and using the relations $ad - bc = 1$, $\alpha\delta - \beta\gamma = 1$, we find

$$\nu = \frac{\left(\dfrac{a}{c} - \dfrac{\alpha}{\gamma}\right)\left(-\dfrac{d}{c} + \dfrac{\delta}{\gamma}\right) + \dfrac{1}{c^2} + \dfrac{1}{\gamma^2}}{\left(\dfrac{a}{c} + \dfrac{d}{c}\right)\left(\dfrac{\alpha}{\gamma} + \dfrac{\delta}{\gamma}\right)} = \mu + \epsilon''.$$

We have then

$$\lambda = \nu - \epsilon' - \epsilon'', \quad |\lambda - \nu| < \epsilon.$$

Since ν is real, the imaginary part of λ cannot exceed ϵ in absolute value, where ϵ is arbitrarily small. It follows that λ is real, and P lies on K.

Each transformation of the group carries the set of limit points into itself. Hence each transformation carries K into itself.

It remains to show that each transformation carries the interior of K into itself. Now every non-loxodromic transformation carries the interior of any of its fixed circles into itself with the exception of the elliptic transformation of period two $(a+d=0)$ with its two fixed points on the fixed circle. We shall show that no transformations of this type occur.

Let T be such a transformation and let I be its isometric circle. Since T carries the interior of I into its exterior there are limit points both within and without I. Let P_1 and P_2 be limit points outside I; and let S be a transformation such that its isometric circle I_s lies near P_1 and that of its inverse I_s' is near P_2, and such that $I, I_s,$ and I_s' are exterior to one another.

We consider the transformation

$$U = ST, \quad U^{-1} = T^{-1}S^{-1} = TS^{-1}.$$

Applying the lemma we have that the isometric circles I_u and I_u' lie respectively in I and I_s' and so are exterior to one another. Both S and U are by hypothesis non-loxodromic and are hence hyperbolic, since the isometric circles of the transformation and its inverse are exterior to one another. Both then carry the interior of K into itself. But this is impossible. Since T carries the interior of K into its exterior and S carries the exterior of K into the exterior, ST carries the interior into the exterior. This contradiction rules out the existence of the elliptic transformation T; and the theorem is established.

4. **Schottky sub-groups.** A Schottky group is a group generated by linear transformations $T_1, \cdots, T_n$, where T_k carries a circle Q_k into a circle Q_k' in such a way that the exterior of Q_k goes into the interior of Q_k' and where the $2n$ circles $Q_1, Q_1', \cdots, Q_n, Q_n'$ are exterior to one another. The limit points of the group lie in the $2n$ circles and form a discrete set of points.

THEOREM 4. *Given a non-elementary properly discontinuous group* Γ, *and a positive constant* ϵ. *Then there exists a Schottky sub-group* Γ' *of* Γ *such that every limit point of* Γ *is within a distance* ϵ *of a limit point of* Γ'.

It is understood here, as stated in the first paragraph of this paper, that the point at infinity is an ordinary point. Let the plane be divided into squares of side $\epsilon/3$ by rulings parallel to the real and imaginary axes. Then only a finite number N of these squares have limit points of Γ within them or on their boundaries. Let $P_1, P_1', \cdots, P_n, P_n'$ be distinct limit points of Γ so chosen that each of the N squares has one at least of the points within it or on its boundary. With $P_1, P_1', \cdots, P_n, P_n'$ as centers let circles C_1, $C_1', \cdots, C_n, C_n'$ respectively be drawn with radii sufficiently small that each circle is exterior to all the others, each radius in any case not exceeding $\epsilon/3$.

Let T_k be a transformation of Γ whose isometric circle I_k lies in C_k and the isometric circle I_k' of whose inverse lies in C_k' (Theorem 1). Then the circles $I_1, I_1', \cdots, I_n, I_n'$ are exterior to one another; and the transformations $T_1, \cdots, T_n$ generate a Schottky group.

Any limit point P of Γ is within a distance $2^{1/2} \epsilon/3$ of one of the points $P_1, \cdots, P_n'$ and each of these latter points is within a distance $\epsilon/3$ of limit points of Γ'. Hence P is within a distance ϵ of limit points of Γ'.

We note that the limit points of Γ' are also limit points of Γ.

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